离散数学（011122）

同济经管 TONGJI SEM

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- 4.1 Definition and Representation of Relations
- 4.2 Relational Operations
- 4.3 Properties of Relations
- 4.4 Equivalence Relations and Partial Order Relations

- 4.1.1 Ordered Pairs and Cartesian Product
- 4.1.2 Definition of Binary Relations
- 4.1.3 Representation of Binary Relations

Definition 4.1- Ordered Pairs

A pair of elements ***x*** and ***y*** arranged in a specific order is called an ***ordered pair*** (or sequence), denoted as $\langle x,y \rangle$

e.g. **Example:**

The Cartesian coordinates of a point: (3,−4)

Properties of Ordered Pairs

- **Order** $<x,y> \neq <y,x>$ （**when** $x \neq y$）
- The necessary and sufficient condition for $<x,y>=<u,v>$ **is** $<x,y>=<u,v> \Leftrightarrow x=u \wedge y=v$

e.g. **Example：** $<2,x+5>=<3y-4,y>$， **solve** x, y.

Solve： $3y-4=2,\ x+5=y \Rightarrow y=2,\ x=-3$

Definition 4.2- Cartesian product

Let A and B be sets, the Cartesian product of A and B is denoted as A×B,

$A \times B = \{ \langle x,y \rangle \mid x \in A \wedge y \in B \}$.

e.g. **Example:** $A=\{0, 1\}$, $B=\{a, b, c\}$

$A \times B=\{\langle 0,a\rangle,\langle 0,b\rangle,\langle 0,c\rangle,\langle 1,a\rangle,\langle 1,b\rangle,\langle 1,c\rangle\}$

$B \times A =\{\langle a,0\rangle,\langle b,0\rangle,\langle c,0\rangle,\langle a,1\rangle,\langle b,1\rangle,\langle c,1\rangle\}$

↳ Properties of the Cartesian Product

- Empty Set Interaction： If either A or B is an empty set, then A × B is the empty set. $A\times\varnothing=\varnothing\times B=\varnothing$
- Not suitable for the commutative property：

  $A\times B\neq B\times A \quad (A\neq B,\ A\neq\varnothing,\ B\neq\varnothing)$
- Not suitable for the associative property：

  $(A\times B)\times C\neq A\times(B\times C) \quad (A\neq\varnothing,\ B\neq\varnothing,\ C\neq\varnothing)$
- The union and intersection operations satisfy the distributive property.

  $A\times(B\cup C)=(A\times B)\cup(A\times C)$

  $(B\cup C)\times A=(B\times A)\cup(C\times A)$

  $A\times(B\cap C)=(A\times B)\cap(A\times C)$

  $(B\cap C)\times A=(B\times A)\cap(C\times A)$
- Cardinality Multiplication Property：

  If $|A|=m$, $|B|=n$, then $|A\times B|=mn$

- **Definition 4.3：** n-dimensional Cartesian product

  (1) An ordered n-tuple is formed by arranging n elements $x_1, x_2, \ldots,$ in a specific order, denoted as $\langle x_1, x_2, \ldots, x_n \rangle$

  (2) Let $A_1, A_2, \ldots,$ be sets., Then the Cartesian product

  $A_1 \times A_2 \times \cdots \times A_n = \{\langle x_1, x_2, \ldots, x_n \rangle \mid x_i \in A_i, i = 1, 2, \ldots, n\}$

  is called the ***n-dimensional***（n-ary） ***Cartesian product***.

e.g.>>>**Example：**

(1,1,0) is the Cartesian coordinate of a point in space,

$(1,1,0) \in \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}$(1, 1, 0).

Definition 4.4- Binary Relations

A set is called a ***binary relation***, denoted as ***R***, if it satisfies one of the following conditions:

(1) The set is non-empty, and its elements are ordered pairs.

(2) The set is empty.

- For example, if $\langle x,y\rangle \in R$, it can be written as $x\,R\,y$; if $\langle x,y\rangle \notin R$, it can be written as $x \not{R}\, y$

e.g. Example:

$R=\{\langle 1,2\rangle,\langle a,b\rangle\}$, $S=\{\langle 1,2\rangle,a,b\}$

- ***R*** is a binary relation, but ***S*** is not a binary relation because *a* and *b* are not ordered pairs.
- Using the above notation, we can write $1R2$, aRb, $a \not{R}\, c$, etc.

Example:

(1) $R=\{<x,y> \mid x,y\in N, x+y<3\}$
$=\{<0,0>, <0,1>, <0,2>, <1,0>, <1,1>, <2,0>\}$

(2) $C=\{<x,y> \mid x,y\in R, x^2+y^2=1\}$,
Where *R* represents the set of real numbers, *C* is the relation between the horizontal and vertical coordinates of points in the Cartesian coordinate plane, and all points in *C* exactly form the unit circle in the coordinate plane.

(3) $R=\{<x,y,z> \mid x,y,z\in R, x+2y+z=3\}$,
R represents a plane in the 3D Cartesian coordinate system.

The employee payroll (relation) is a set of ***5-tuples representing*** employees:**<301,B,25,M,19000>,<302,O,23,F,18500>**

ID No.	Name	Age	Gender	Salary
301	B	25	M	19000
302	O	23	F	18500
303	Y	37	M	25000
304	M	31	M	22000
…	…	…	…	…

↳ Binary relation from A to B & Binary relation on A

Definition 4.5

Let *A* and *B* be sets. Any subset of ***A*×*B*** that defines a binary relation is called a ***binary relation from A to B***. When ***A*=*B***, it is called a ***binary relation on A***.

Counting:

- $|A|=n$, $|B|=m$, $|A\times B|=nm$, and there are 2^{nm} subsets of ***A*×*B***. Therefore, there are 2^{nm} different binary relations from *A* to *B*.
- If $|A|=n$, there are 2^{n^2} different binary relations on *A*.
- For example, if $|A|=3$, then there are 512 different binary relations on *A*.

Note: Sets are the static carriers of relations, while relations are the dynamic interaction rules between sets.

- Let A be any set. The empty set ∅ is considered as the ***empty relation*** on A.
- **Definition 4.6:**
  - E_A is called the ***universal relation*** on A, where
    $E_A=\{<x,y> \mid x \in A \wedge y \in A\}=A \times A$
  - I_A is called the ***identity relation*** on A, where
    $I_A=\{<x,x> \mid x \in A\}$

e.g. **Example: $A=\{1,2\}$, then**

$E_A=\{<1,1>,<1,2>,<2,1>,<2,2>\}$

$I_A=\{<1,1>,<2,2>\}$

Definition 4.7:

The less than or equal to relation L_A, the *divides relation* D_B, and the *containment relation* $R_{\subseteq}$ are defined as follows:

- $L_A=\{<x,y>| \ x,y\in A \wedge x\leq y\}$, $A\subseteq R$, R is the set of real numbers.
- $D_B=\{<x,y>| \ x,y\in B \wedge x$ divides $y\}$, $B\subseteq Z^*$, Z^* is the set of non-zero integers.
- $R_{\subseteq}=\{<x,y>| \ x,y\in A \wedge x\subseteq y\}$, where A is a family of sets.

e.g. **Example:** $A=\{1,2,3\}$, $B=\{a,b\}$, then

$$L_A=\{<1,1>,<1,2>,<1,3>,<2,2>,<2,3>,<3,3>\}$$

$$D_A=\{<1,1>,<1,2>,<1,3>,<2,2>,<3,3>\}$$

e.g. **Example:** $B=\{a,b\}$, $A=P(B)=\{\varnothing,\{a\},\{b\},\{a,b\}\}$,

then the ***inclusion relation on*** A is :

$$R_\subseteq=\{<\varnothing,\varnothing>,<\varnothing,\{a\}>,<\varnothing,\{b\}>,<\varnothing,\{a,b\}>,<\{a\},\{a\}>,<\{a\},\{a,b\}>,<\{b\},\{b\}>,<\{b\},\{a,b\}>,<\{a,b\},\{a,b\}>\}$$

Similarly, greater than or equal to relations, less than relations, greater than relations, proper inclusion relations, etc., can also be defined.

- **Ways to Represent Relations:**

  Set-theoretic expression of relations; Relational matrix; Relational graph

- **Definition 4.8: Relational Matrix**

  If $A=\{x_1, x_2, \ldots, x_m\}$ and $B=\{y_1, y_2, \ldots, y_n\}$ and R is a relation from A to B, then the ***relational matrix*** of R is the Boolean matrix $M_R = [r_{ij}]_{m\times n}$, where

  $r_{ij} = 1 \Leftrightarrow < x_i, y_j > \in R$.

- **Definition 4.9: Relational Graph**

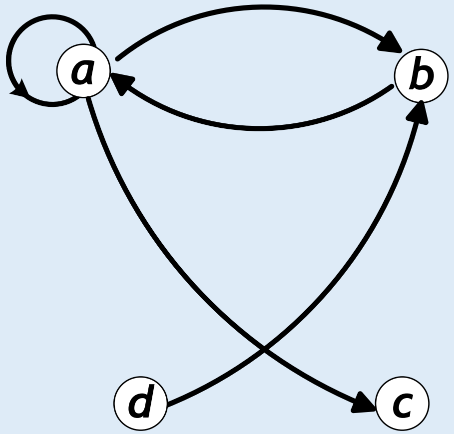
  If $A= \{x_1, x_2, \ldots, x_m\}$, and R is a relation on A, ***the relational graph*** of R ***is*** $G_R=<A, R>$, where A is the set of vertices and R is the set of edges. **If** $<x_i,x_j>$ **belongs to** R, there is a directed edge from x_i to x_j in the graph.

Note: Let A and B be finite sets.

- **Relational matrices** are suitable for representing relations from A to B or relations on A.
- **Relational graphs** are suitable for representing relations on A.

e.g. **Example:** $A=\{a, b, c, d\}$, $R=\{\langle a, a\rangle, \langle a, b\rangle, \langle a, c\rangle, \langle b, a\rangle, \langle d, b\rangle\}$,
The relationship matrix M_R **and** the relationship graph G_R

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



Objective：

Key Concepts：

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4.2.1 Basic Operations of Relations

- Domain, range, domain (again), inverse, composition
- Properties of basic operations

4.2.2 Power Operations of Relations

- Definition of power operations
- Methods of power operations
- Properties of power operations